AD-A117 072 TEXAS A AND M UNIV COLLEGE STATION INST OF STATISTICS F/6 12/1
SIMULTANEOUS CONFIDENCE BANDS FOR AUTOREGRESSIVE SPECTRA.(U)
JUN 82 H J NEWTON, M PAGANO N00014-82-MP-2001
UNCLASSIFIED TR-N-32 NL

1 OF 1
AD A
117072

END
DATE
FILMED
08-82
DTIC

(12)

TEXAS A&M UNIVERSITY
COLLEGE STATION, TEXAS 77843-3143

'UTE OF STATISTICS
713 - 845-3141

AD A117072

SIMULTANEOUS CONFIDENCE BANDS
FOR AUTOREGRESSIVE SPECTRA

H. Joseph Newton
Institute of Statistics
Texas A&M University

Marcello Pagano
Department of Biostatistics
Harvard University

Technical Report No. N-32

June 1982

Texas A&M Research Foundation
Project No. 4226T

"Multiple Time Series Modeling and
Time Series Theoretic Statistical Methods:

Sponsored by the Office of Naval Research

Professor Emanuel Parzen, Principal Investigator

007

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER N-32	2. GOVT ACCESSION NO. AD-A117072	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) SIMULTANEOUS CONFIDENCE BANDS FOR AUTOREGRESSIVE SPECTRA		5. TYPE OF REPORT & PERIOD COVERED Technical
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) H. Joseph Newton and Marcello Pagano		8. CONTRACT OR GRANT NUMBER(s) ONR N00014-82-MP-2001 ARU DAAG 29-80-C-0070
9. PERFORMING ORGANIZATION NAME AND ADDRESS Texas A&M University Institute of Statistics College Station, TX 77843		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS		12. REPORT DATE June 1982
		13. NUMBER OF PAGES
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

NA

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Spectral density function; autoregressive spectral estimation, Scheffé projections.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Asymptotic simultaneous confidence bands for an autoregressive spectral density are derived and illustrated.

DD FORM 1473 1 JAN 73 EDITION OF 1 NOV 65 IS OBSOLETE

S/N 0102-LF-014-6601

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

1. INTRODUCTION

Autoregressive spectral estimation has become an important method of spectral estimation in recent years (see Akaike (1969), Parzen (1974), Ulrych and Bishop (1975), and Beamish and Priestley (1981), for example) despite 1) a continuing discussion of the problems of order determination and 2) a lack of easily applied procedures for determining confidence intervals or bands on the function being estimated.

In this paper we derive asymptotic 100 $(1-\alpha)\%$ simultaneous confidence bands for an autoregressive spectral density assuming one has data from a finite order autoregressive process with known order or conditional upon having correctly determined the order if it is unknown.

The bands are derived in section 2 and their implementation described on simulated data in section 3.

2. SIMULTANEOUS CONFIDENCE BANDS

The basic property of autoregressive processes that we shall use is that the reciprocal of the autoregressive spectral estimator is a linear combination of a finite number of asymptotically normal random variables whose asymptotic covariance matrix is easily consistently estimated. Thus one can use a Scheffé (1953) type projection argument to determine asymptotic confidence bands on the reciprocal of the autoregressive spectral density (and thus on the density itself) at all frequencies.

Let Y be an autoregressive process of order p with coefficients $\alpha(1),\ldots,\alpha(p)$ and noise variance σ^2, i.e.

$$\sum_{j=0}^{p} \alpha(j)Y(t-j) = \varepsilon(t) \quad , \quad t=0,\pm 1,\pm 2,\ldots$$

where $\alpha(0) = 1$, ε is a white noise series of uncorrelated zero mean random variables having common variance σ^2, and the zeros of the complex valued polynomial $g(z) = \sum_{j=0}^{p} \alpha(j)z^j$ are all greater than one in modulus. Then we write $Y \sim AR(p,\underset{\sim}{\alpha},\sigma^2)$ and note that the covariance function $R(v) = E(Y(t)Y(t+v))$, $v=0,\pm 1,\ldots$, the spectral density function $f(\omega) = \frac{1}{2\pi} \sum_{v=-\infty}^{\infty} R(v)e^{-iv\omega}$, $\omega\varepsilon[-\pi,\pi]$, and the parameters $\underset{\sim}{\alpha}$ and σ^2 are related by

$$f(\omega) = \frac{\sigma^2}{2\pi} \frac{1}{|g(e^{i\omega})|^2} \quad , \quad \omega\varepsilon[-\pi,\pi], \tag{2.1}$$

$$\sum_{j=0}^{p} \alpha(j)R(j-v) = \delta_v \, \sigma^2 \quad , \quad v\geq 0, \tag{2.2}$$

where δ_v is the Kronecker delta. If on the other hand, Y is a moving average process of order q and parameters $\beta(1),\ldots,\beta(q)$, and σ^2, i.e

$$Y(t) = \sum_{k=0}^{q} \beta(k)\varepsilon(t-k) \quad , \quad t=0\pm 1,\ldots,$$

where $\beta(0) = 1$, we have, with $h(z) = \sum_{k=0}^{q} \beta(k)\, z^k$,

$$R(v) = \begin{cases} \sigma^2 \sum_{k=0}^{q-|v|} \beta(k)\beta(k+|v|) & , \quad |v|\leq q \\ 0 & , \quad |v| > q \end{cases} ,$$

$$f(\omega) = \frac{1}{2\pi} \sum_{v=-q}^{q} R(v)e^{-iv\omega}$$

$$= \frac{1}{2\pi} [R(0) + 2 \sum_{v=1}^{q} R(v) \text{ Cos } v\omega]$$

$$= \frac{\sigma^2}{2\pi} |h(e^{i\omega})|^2 \quad , \ \omega\varepsilon[-\pi,\pi].$$

Thus, if $Y \sim AR(p,\underset{\sim}{\alpha},\sigma^2)$ we have $f(\omega)>0$ for all ω and

$$h(\omega) = \frac{1}{f(\omega)} = \frac{4\pi^2/\sigma^2}{2\pi} \quad |g(e^{i\omega})|^2 = \underset{\sim}{x}^T(\omega)\underset{\sim}{\gamma}$$

where $\underset{\sim}{x}^T(\omega) = (\frac{1}{2\pi}, \frac{1}{\pi} \text{ Cos } \omega, \ldots, \frac{1}{\pi} \text{ Cos } p\omega)$, and $\underset{\sim}{\gamma}^T = (\gamma(0),\gamma(1),\ldots,\gamma(p))$, with

$$\gamma(v) = \frac{4\pi^2}{\sigma^2} \sum_{j=0}^{p-v} \alpha(j)\alpha(j+v) \quad , \quad v=0,\ldots,p.$$

Thus the reciprocal of f is a linear combination of the parameters $\gamma(0),\gamma(1),\ldots,\gamma(p)$.

Let $Y \sim AR(p,\underset{\sim}{\alpha},\sigma^2)$ and let $Y(1),\ldots,Y(n)$ be a sample realization from Y. Let $\hat{\underset{\sim}{\theta}}^T = (\hat{\underset{\sim}{\alpha}}^T,\hat{\sigma}^2) = (\hat{\alpha}(1),\ldots,\hat{\alpha}(p),\hat{\sigma}^2)$ be estimators of the parameters $\underset{\sim}{\theta}^T = (\underset{\sim}{\alpha}^T,\sigma^2) = (\alpha(1),\ldots,\alpha(p),\sigma^2)$ that have the same asymptotic distribution as the maximum likelihood estimators of $\underset{\sim}{\theta}$, i.e (see Parzen (1961))

$$\sqrt{n} \quad (\hat{\underset{\sim}{\theta}}-\underset{\sim}{\theta}) \xrightarrow{\mathcal{D}} N_{p+1}(0,\Sigma(\underset{\sim}{\theta})) \tag{2.3}$$

where

$$\underset{\sim}{\ddagger}(\underset{\sim}{\theta}) = \begin{bmatrix} \sigma^2\Gamma_p^{-1} & \underset{\sim}{0} \\ \underset{\sim}{0}^T & 2\sigma^4 \end{bmatrix} ,$$

and Γ_p is the $(p\times p)$ Toeplitz covariance matrix of $Y(1),\ldots,Y(p)$. Estimators having this asymptotic distribution include

(1) The Yule-Walker estimators (YWE), i.e the solution to the Yule-Walker equations (2.2) with

$$\hat{R}(v) = \frac{1}{T}\sum_{t=1}^{T-v} Y(t)Y(t+v) \quad , \quad v=0,\ldots,p$$

replacing $R(v)$.

(2) The Least Squares estimators (LSE), i.e the values $\hat{\alpha}(1),\ldots,\hat{\alpha}(p)$ minimizing

$$S(\underset{\sim}{\alpha}) = \sum_{t=p+1}^{T} \{Y(t)-\alpha(1)Y(t-1)-\ldots-\alpha(p)Y(t-p)\}^2,$$

and $\hat{\sigma}^2 = S(\hat{\underset{\sim}{\alpha}})/(n-p)$.

3) The Burg estimators (BGE), (see Ulrych and Bishop (1975) for a description of these estimators).

Then the autoregressive spectral estimator $\hat{f}$ of f consists of evaluating (2.1) with $\hat{\underset{\sim}{\theta}}$ replacing $\underset{\sim}{\theta}$. If the order p is unknown a priori there are several procedures (see Hannan and Quinn (1979), for example) for finding a value $\hat{p}$ from the data which as $n\to\infty$ has probability one of determining the correct order p.

Then the confidence bands are given by the following Theorem:

<u>Theorem 2.1</u> Let $Y(1),\ldots,Y(n)$ be a sample realization from a Gaussian $AR(p,\underset{\sim}{\alpha},\sigma^2)$ time series and let $\hat{\alpha}(1),\ldots,\hat{\alpha}(p)$, and $\hat{\sigma}^2$ be estimators satisfying (2.3). Let

$$\hat{\gamma}(v) = \frac{4\pi^2}{\hat{\sigma}^2} \sum_{j=0}^{p-v} \hat{\alpha}(j)\hat{\alpha}(j+v) \quad , \quad v=0,\ldots,p$$

and $\hat{h}(\omega) = \dfrac{1}{\hat{f}(\omega)} = \dfrac{1}{2\pi}\,[\hat{\gamma}(0) + 2\sum_{v=1}^{p} \hat{\gamma}(v)\ \mathrm{Cos}\ v\omega]$. Let $\underset{\sim}{\beta}^T = (\underset{\sim}{\alpha}^T,\ 1/\sigma^2)$, $\hat{\underset{\sim}{\beta}}^T = (\hat{\underset{\sim}{\alpha}}^T,\ 1/\hat{\sigma}^2)$, and $B(\underset{\sim}{\beta})$ be the $(p+1)\times(p+1)$ matrix having (ℓ,m) element

$$B_{\ell m}(\underset{\sim}{\beta}) = \begin{cases} \sigma^2\gamma(\ell-1) & , \quad m=p+1 \\ \dfrac{4\pi^2}{\sigma^2}\,\delta_1(\ell,m) + \dfrac{4\pi^2}{\sigma^2}\,\delta_2(\ell,m) & , \quad m=1,\ldots,p \end{cases}$$

$\ell=1,\ldots,p+1$, where $\delta_1(\ell,m) = \alpha(m+(\ell-1))$ if $m\leq p-(\ell-1)$ and zero otherwise, while $\delta_2(\ell,m) = \alpha(m-(\ell-1))$ if $m\geq\ell-1$ and zero otherwise. Then:

a) $\sqrt{n}\ (\hat{\underset{\sim}{\gamma}}-\underset{\sim}{\gamma}) \xrightarrow{\mathcal{D}} N_{p+1}\ (\underset{\sim}{0},\ddagger\ (\underset{\sim}{\gamma}))$ where $\ddagger(\underset{\sim}{\gamma}) = B(\underset{\sim}{\beta})\ddagger(\underset{\sim}{\beta})B(\underset{\sim}{\beta}^T)$, and

$$\ddagger(\underset{\sim}{\beta}) = \begin{bmatrix} \sigma^2\Gamma_p^{-1} & \underset{\sim}{0} \\ \underset{\sim}{0}^T & 2/\sigma^4 \end{bmatrix}$$

b) Asymptotically, as $n\to\infty$, the probability is at least $1-\alpha$ that simultaneously for all $\omega \in [0,\pi]$,

$$\frac{1}{\hat{h}(\omega) + S(\omega)} \leq f(\omega) \leq \frac{1}{\hat{h}(\omega) - S(\omega)}$$

where if $\hat{h}(\omega) - S(\omega) \leq 0$ we use infinity as the upper limit, and

$$S^2(\omega) = \frac{\chi^2_{\alpha,p+1}}{n} \; \underset{\sim}{x}^T(\omega)\ddagger(\hat{\underset{\sim}{\gamma}})\underset{\sim}{x}(\omega),$$

with $\chi^2_{\alpha,p+1}$ being the upper α critical value of a Chi square distribution having p+1 degrees of freedom.

<u>Proof</u> To prove (a) we note that from (2.3) and the definition of $\underset{\sim}{\beta}$ we have $\sqrt{n}\,(\hat{\underset{\sim}{\beta}}-\underset{\sim}{\beta}) \xrightarrow{\mathcal{D}} N_{p+1}(\underset{\sim}{0},\ddagger(\underset{\sim}{\beta}))$. Then defining $c = 1/\sigma^2$ and $\hat{c} = 1/\hat{\sigma}^2$, we have for $v=0,\ldots,p$,

$$\sqrt{n}\,[\hat{\gamma}(v) - \gamma(v)] = 4\pi^2 \sum_{j=0}^{p-v} \sqrt{n}\,\{\hat{c}\hat{\alpha}(j)\hat{\alpha}(j+v) - c\alpha(j)\alpha(j+v)\}$$

$$= 4\pi^2 \sum_{j=0}^{p-v} \sqrt{n}\,\{\hat{c}\hat{\alpha}(j)\hat{\alpha}(j+v) - \hat{c}\alpha(j)\hat{\alpha}(j+v) + \hat{c}\alpha(j)\hat{\alpha}(j+v)$$

$$- c\alpha(j)\hat{\alpha}(j+v) + c\alpha(j)\hat{\alpha}(j+v) - c\alpha(j)\alpha(j+v)\}$$

$$= 4\pi^2 \sum_{j=0}^{p-v} \sqrt{n}\,\{\hat{c}\hat{\alpha}(j+v)[\hat{\alpha}(j)-\alpha(j)] + \alpha(j)\hat{\alpha}(j+v)(\hat{c}-c)$$

$$+ c\alpha(j)[\hat{\alpha}(j+v) - \alpha(j+v)]\}$$

$$\approx 4\pi^2 \sum_{j=0}^{p-v} \{c\alpha(j+v)\,\sqrt{n}\,[\hat{\alpha}(j)-\alpha(j)] + \alpha(j)\alpha(j+v)\,\sqrt{n}\,(\hat{c}-c)$$

$$+ c\alpha(j)\,\sqrt{n}\,[\hat{\alpha}(j+v) - \alpha(j+v)]\}$$

$$= 4\pi^2 \sum_{j=0}^{p-v} c\alpha(j+v)\,\sqrt{n}\,[\hat{\alpha}(j)-\alpha(j)]$$

$$+ 4\pi^2\sqrt{n}\,(\hat{c}-c) \sum_{j=0}^{p-v} \alpha(j)\alpha(j+v) + 4\pi^2 \sum_{j=v}^{p} c\alpha(j-v)\,\sqrt{n}\,[\hat{\alpha}(j)-\alpha(j)],$$

where we write $X_n \overset{\mathcal{D}}{\sim} Y_n$ to indicate that the sequences $\{X_n\}$, $\{Y_n\}$ of random variables converge in distribution to the same random variables, and we have used Slutsky's Theorem (Rao (1973), p. 122) repeatedly.

Thus we have shown that $\sqrt{n}\,(\hat{\gamma}-\gamma) \overset{\mathcal{D}}{\sim} \sqrt{n}\, B(\beta)(\hat{\beta}-\beta)$ and (a) follows.

From (a) then we have

$$n(\hat{\gamma}-\gamma)^T \, \Sigma^{-1}(\gamma) \; (\hat{\gamma}-\gamma) \xrightarrow{\mathcal{D}} \chi^2_{p+1}$$

and since the elements of $\Sigma(\gamma)$ are continuous functions of the elements of γ we have (Rao (1973), p. 388)

$$n(\hat{\gamma}-\gamma)^T \, \Sigma^{-1} \; (\hat{\gamma})(\hat{\gamma}-\gamma) \xrightarrow{\mathcal{D}} \chi^2_{p+1} \quad .$$

Thus the probability is (asymptotically) $1-\alpha$ that the true parameter γ lies inside the ellipsoid defined as the set of vectors ℓ satisfying

$$(\hat{\gamma}-\ell)^T \, M(\hat{\gamma}-\ell) \leq 1 \quad ,$$

where $M = n\,\Sigma^{-1}(\hat{\gamma})/\chi^2_{\alpha,p+1}$ and $\chi^2_{\alpha,p+1}$ is the upper α critical value of a χ^2_{p+1} distribution. But (Scheffe (1959), p. 407, et. sec.), γ is in this ellipsoid if and only if

$$|x^T \; (\hat{\gamma}-\gamma)| \leq [x^T \, M^{-1} x]^{\frac{1}{2}}$$

for all (p+1) dimensional vectors x and thus in particular only if (for vectors $x(\omega)$)

$$|\hat{h}(\omega)-h(\omega)| \le S(\omega) \quad , \qquad [0,\pi],$$

giving a $100(1-\alpha)\%$ simultaneous confidence band for $h(\omega) = 1/f(\omega)$ as

$$\hat{h}(\omega) - S(\omega) \le h(\omega) \le \hat{h}(\omega) + S(\omega).$$

Then the reciprocal of this gives a simultaneous confidence band for f, where if $\hat{h}(\omega) - S(\omega) < 0$ we use infinity as the upper limit on $f(\omega)$ which we can do without decreasing the probability content of the bands.

3. EXAMPLES

To illustrate the method of section 2 we calculated the autoregressive spectral estimator and confidence bands for 10 sample realizations of length 200 for each of three AR processes (see Beamish and Priestley (1981)); i.e AR(2, -.4, -.45, 1), AR(5, 1.7, 2.4, 1.634, .872, .168, 1), and AR(4, -2.7607, 3.8106, -2.6535, .9238, 1). These processes have characteristic polynomial zeros whose moduli are given by (1.11, 2.00), (1.12, 1.12, 1.19, 1.19, 3.33), and (1.01998, 1.01998, 1.019798, 1.019798) respectively, and have variances 2.66, 30.59, and 761.3. Further, the ratio of the maximum to the minimum values of the spectral densities for the three models are approximately 111.5, 41,800, and 6×10^6. Thus the performance of our method on these models should be representative of its performance on a wide class of AR models. In Figures A-C we display on a log scale typical point estimators and 95% confidence bands for the spectral density of the AR(2), AR(5), and AR(4) models respectively using the Burg estimators. The true

spectral density (the curve with the x's) is superimposed on the graphs. Each curve consists of connecting values of the various functions at the 101 equally spaced points between 0 and π (labelled on the graph 0 to 1/2 cycles per sampling interval).

On the graphs, the largest noninfinite value of upper limits is assigned to $1/(\hat{h}(\omega) - S(\omega))$ whenever $\hat{h}(\omega) - S(\omega) < 0$. Examples of this include frequencies near .12 for the AR(4) model and those around .32 for the AR(5). We have found this to be a useful diagnostic in practice as peaks such as those in these simulated data may in fact be indicating the existence of deterministic components in real data in which case the spectral density does not exist.

In summary then, the confidence bands given in section 2 appear to work well on a wide variety of autoregressive models.

REFERENCES

Akaike, H. (1969), "Fitting Autoregressive Models for Prediction," _Ann. Inst. Statist. Math._, _21_, 243-247.

Beamish, N. and Priestley, M. B. (1981), "A Study of Autoregressive and Window Spectral Estimation," _Appl. Statist._, _30_, 41-58.

Hannan, E. J. and Quinn, B. G. (1979), "The Determination of the Order of an Autoregression," _JRSS B_, _41_, 190-195.

Parzen, E. (1961), "An Approach to Time Series Analysis," _Annals of Math. Statist._, _32_, 383-421.

________. (1974), "Recent Advances in Time Series Modeling," _IEEE Transactions on Automatic Control_, _19_, 723-730.

Rao, C. R. (1973), _Linear Statistical Inference and Its Applications_, 2nd ed., New York: Wiley.

Scheffé, H. (1953), "A Method for Judging all Contrasts in the Analysis of Variance," _Biometrika_, _40_, 87-104.

________. (1959), _The Analysis of Variance_, New York: Wiley.

Ulrych, T. L. and Bishop, T. N. (1975), "Maximum Entropy Spectral Analysis and Autoregressive Decomposition," _Rev. of Geophys. and Space Phys._, _14_, 183-200.

FIGURE A. Typical Point Estimator and 95% Confidence Bands with True Spectra Superimposed for AR(2) model Using Burg Estimator

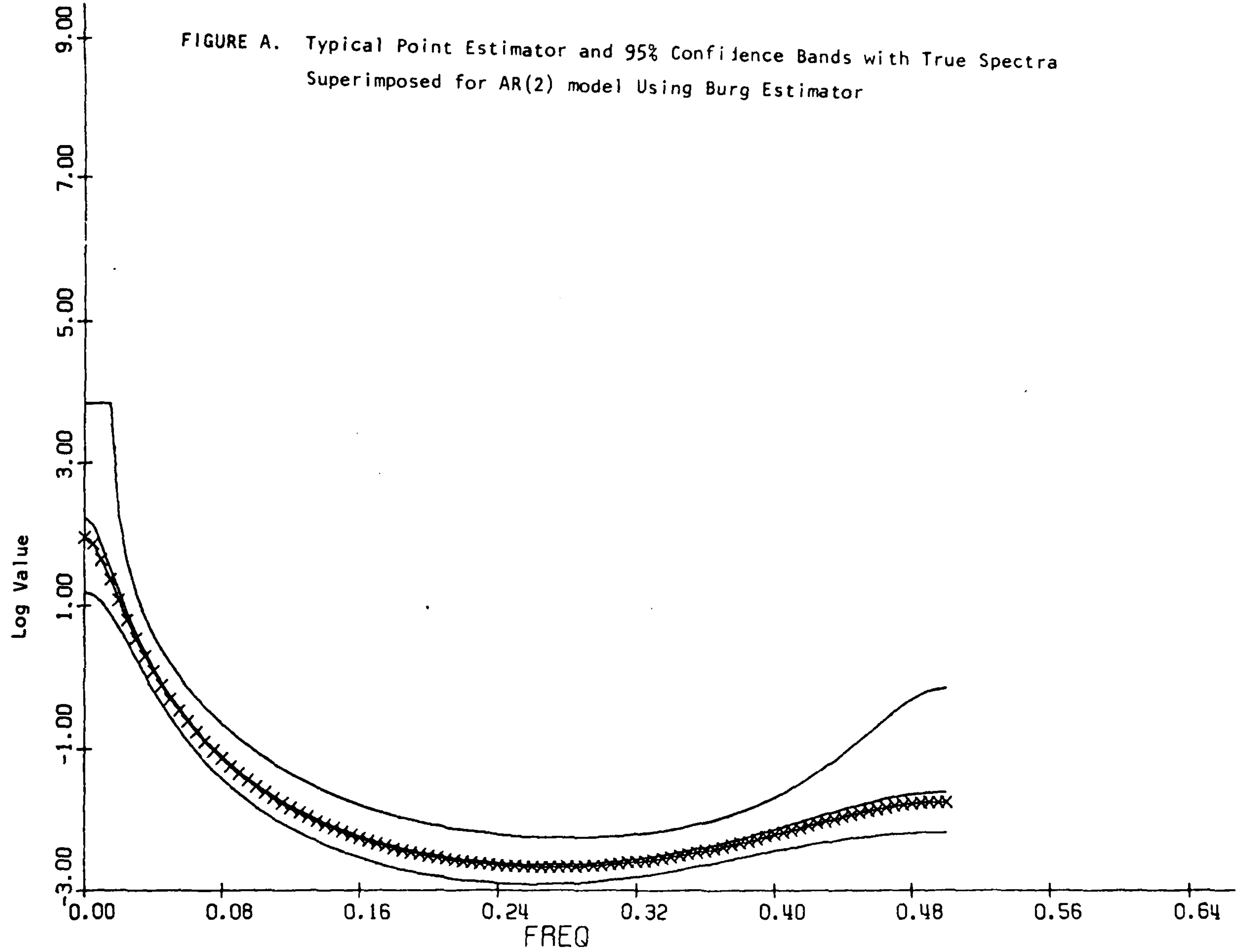
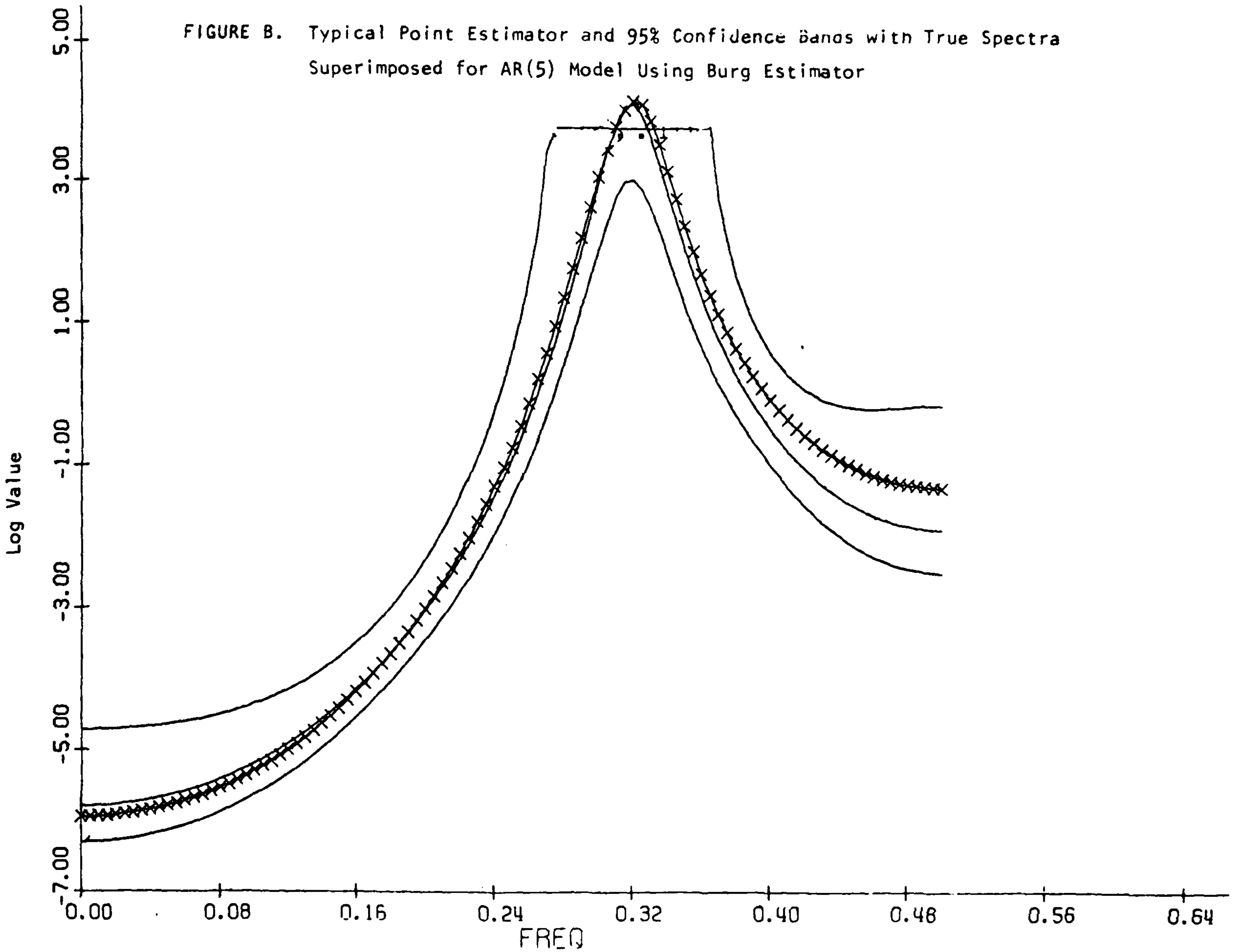


FIGURE B. Typical Point Estimator and 95% Confidence Bands with True Spectra Superimposed for AR(5) Model Using Burg Estimator



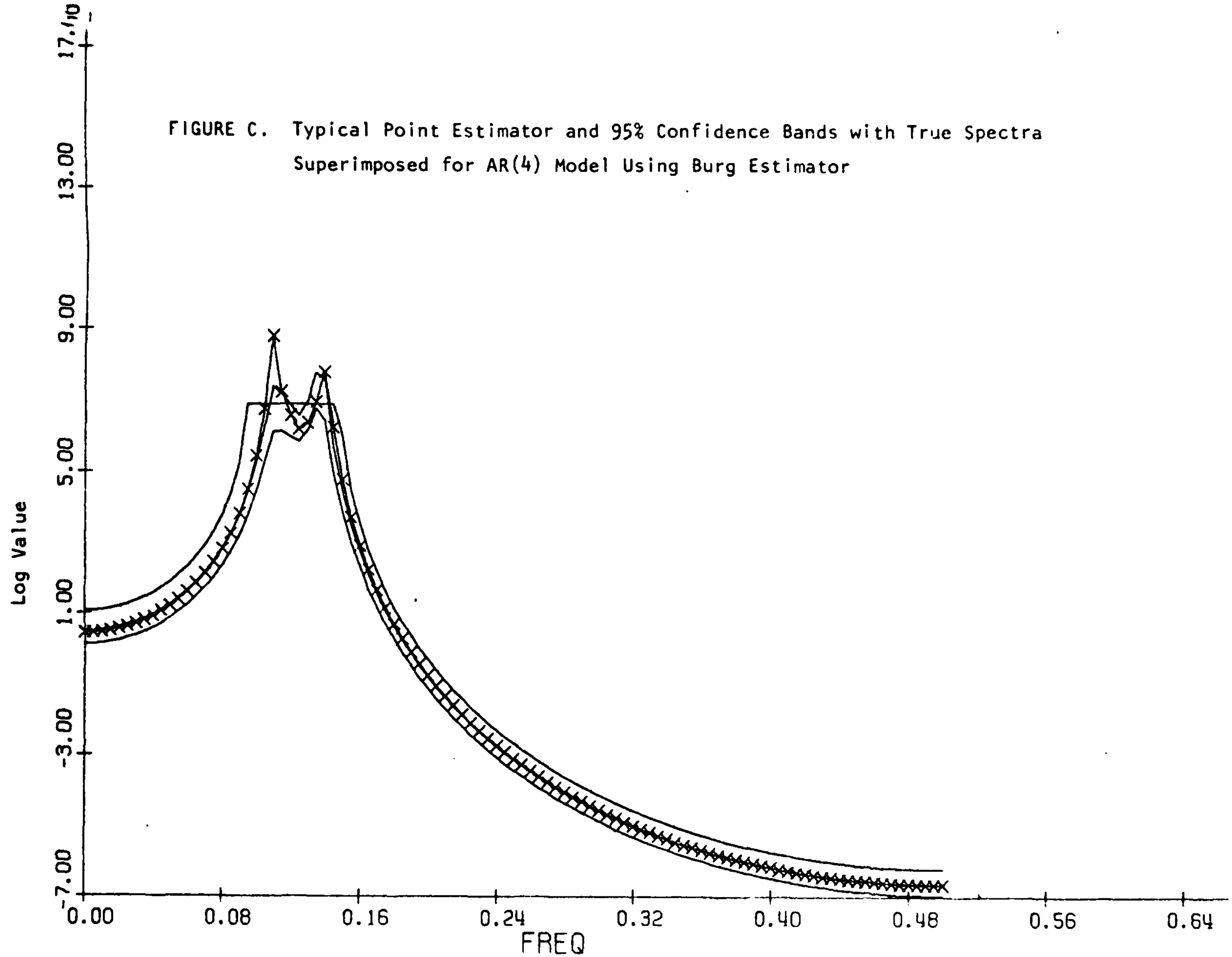


FIGURE C. Typical Point Estimator and 95% Confidence Bands with True Spectra Superimposed for AR(4) Model Using Burg Estimator

FILMED

8-8